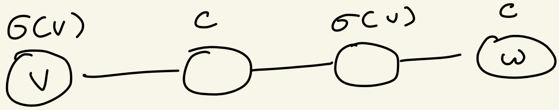
Flip Dynamics

1. $\boxed{\text{Choose } v \in_R V, \quad c \in_R C}$

2. Let $S_\sigma(v, c) = \{ w \mid$ there is an alternating path b/w v, w using colors c and $\sigma(v) \}$

let $S_\sigma(v, \sigma(v)) = \phi$

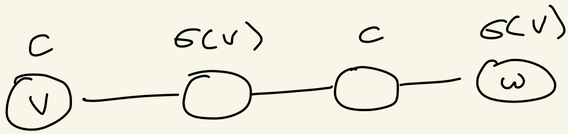
For ex -



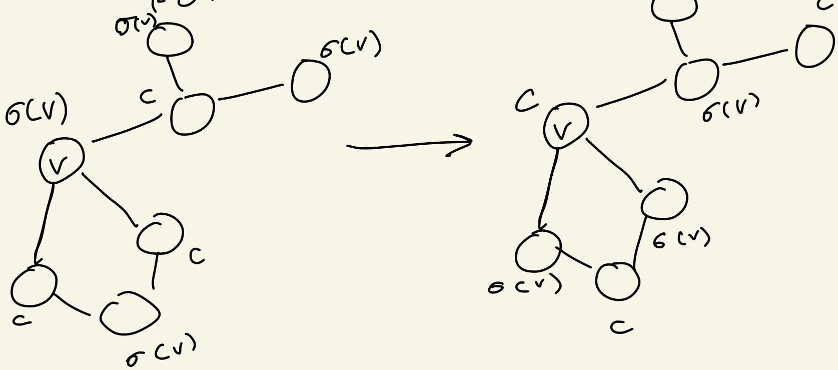
Let $\alpha = | S_\sigma(v, c) |$

With prob $\frac{p_\alpha}{\alpha}$ flip $S_\sigma(v, c)$ by interchanging c and $\sigma(v)$

For ex -



For ex



Why $\frac{p_\alpha}{\alpha}$? Each $u \in S_\sigma(v, c)$ gives rise to the same cluster.

$\Rightarrow$ Each cluster is flipped with prob p_α.

Irreducible: When $c \notin \{\sigma(w) \mid w \in N(v)\}$ (when $|C| \geq \Delta+2$) equivalent to Glauber Dynamics. (which is irreducible).

Aperiodic: Self loops when $c = \sigma(v)$

Symmetric: Flipping $S_\sigma(v,c)$ results in coloring σ', Flipping $S_{\sigma'}(v, \sigma(v))$ recovers σ.
Both occur with prob p_α where $\alpha = |S_\sigma(v,c)| = |S_{\sigma'}(v,\sigma(v))|$

Prob Vales -

$$p_\alpha = \max\left(0, \frac{13}{42} - \frac{1}{7}\left[1 + \frac{1}{2} + \cdots + \frac{1}{\alpha - 2}\right]\right)$$

$$\Rightarrow p_1 = 1, \quad p_2 = \frac{13}{42}, \quad p_3 = \frac{1}{6}$$

$$p_4 = \frac{2}{21}, \quad p_5 = \frac{1}{21}, \quad p_6 = \frac{1}{84}$$

$$p_\alpha = 0, \quad \alpha > 6.$$

Why these? : In the analysis we will need —

i) $2(i-1)p_i + p_{2i+1} \leq \frac{2}{3}$

ii) $(j-1)(p_j - p_{j+1}) + i(p_i - p_{i+1}) < \frac{5}{6}$

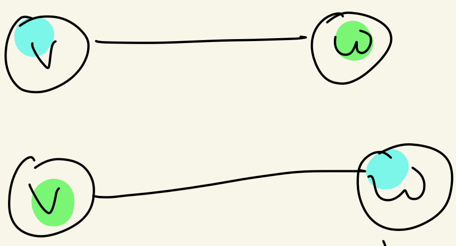
iii) $i p_i \leq p_1 = 1$

iv) $(i-1)p_i \leq 2p_3 = \frac{1}{3}$ v) $(i-c)p_i < \frac{1}{4}$ $\forall\ c \geq 2$

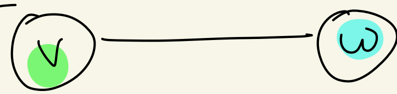
Path Coupling Analysis

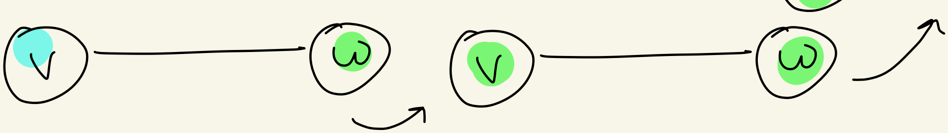
$\phi(\sigma, \tau)$ = Hamming Distance b/w σ and τ

Consider two states that differs at v and w as —

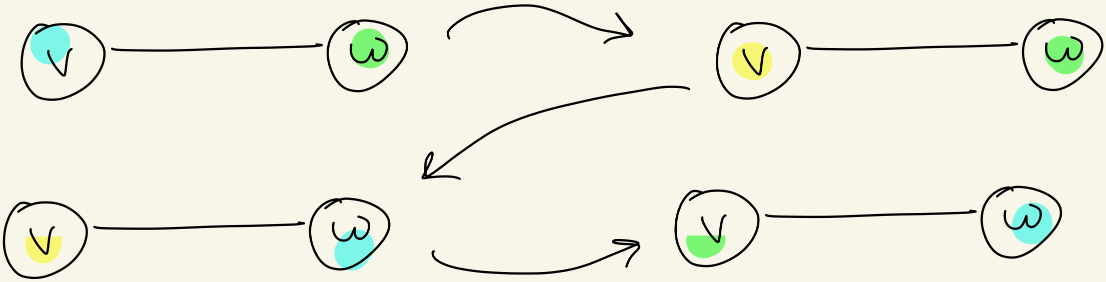


Hamming Distance = 2





Distance in space of valid colorings = 3



Thus we redefine state space to be C^V.

→ If the chain is started at a proper coloring it stays a proper coloring

→ If the chain starts from an improper, it'll eventualy reach a proper

(Doesn't this break irreducibility?)

Simplifying

→ Each $y \in S_\sigma(x, c)$ gives rise to the same cluster.
Thus it is enough to reason about clusters instead of vertices.

When can clusters differ?

Suppose σ, τ differ only at v.

Let $w \in N(v)$

The clusters that are different

i) $S_\sigma(v, c)$ and $S_\tau(v, c)$

$\downarrow$ Has $\sigma(v)$ and c

$\downarrow$ Has $\tau(v)$ and c

ii) $S_\sigma(w, \tau(v))$ and $S_\tau(w, \sigma(v))$

v "blocks" the cluster in σ, whereas it doesn't in τ

For all other clusters, they can either -

i) Be re-indexed to one of the two above cases

or ii) Be exactly the same in both σ and τ.

For the 2nd case, we use the identity coupling and so distance does not change.

Analysis set up

Let

Γ_c = neighbours of v colored c

$$D_c = \left\{ \begin{array}{l} \left\{ S_\sigma(v,c), \; S_\tau(v,c), \right. \\ \left\{ S_\sigma(w,\tau(v)), \; S_\tau(w,\sigma(v)) \right\}_{w \in \Gamma_c} \end{array} \right\}$$

All differing clusters associated with c.

Note: $D_c \cap D_{c'} = \phi$ except for $D_{\sigma(v)} \cap D_{\tau(v)}$ (which may or may not)

For ex -

$D_{\sigma(v)} = \cancel{S_\sigma(v,\sigma(v))}$ ϕ

$s_\tau(v,\sigma(v))$

$s_\sigma(w_i, \tau(v))$

$\cancel{s_\tau(w_i, \sigma(v))}$ ϕ

$w_i \in \delta_{\sigma(v)}$

$D_{\tau(v)} = S_\sigma(v,\tau(v))$

$\cancel{s_\tau(v,\tau(v))}$ ϕ

$\cancel{s_\sigma(w_i, \tau(v))}$ ϕ

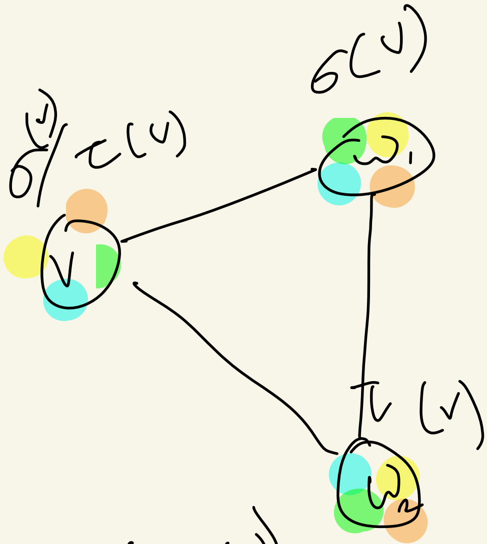
$s_\tau(w_i, \sigma(v))$

$w_i \in \delta_{\tau(v)}$

$\leadsto s_\sigma(w_1, \tau(v)) = s_\tau(w_2, \sigma(v))$

$\to S_\sigma(w_1, \tau(v))$

$\to s_\tau(v, \sigma(v))$



$S_\sigma(v, \tau(v))$

$S_\tau(w_2, \sigma(v))$

This case will be dealt with separately.

our coupling f is such that

(i) If $S \notin \bigcup_c D_c$, $f(S) = S$

Identity.

(ii) If $S \in D_c$, $f(S) \in D_c$

Thus $\mathbb{E}(\Delta \Phi) =$

$$\sum_c \frac{1}{k} \mathbb{E}[\Delta_{D_c} \Phi]$$

expected change in ϕ given both chains flip a cluster in D_c.

eqv to one chain flips a cluster in D_c by (ii)

Let $\delta_c = |\Gamma_c|$ = no. of c colored neighbours.

Main Lemma

a) If $\delta_c = 0$, $\mathbb{E}[\Delta_{D_c}\Phi] \le -1$

b) If $\delta_c > 0$, $\mathbb{E}[\Delta_{D_c}\Phi]$

$$\le \frac{11}{6}\delta_c - 1$$

Main Thm

Flip Dynamics mixes in $O(nk\log n)$ given

$$k > \frac{11\Delta}{6}$$

Proof (given the lemma):

\# colors c, $\delta_c = 0$ (i.e. doesn't appear in neighborhood of v)

$$= k - \left|\{c' \mid \underbrace{\delta_{c'} > 0}_{\text{colors in at least one neighbour.}}\}\right|$$

$$= k - \sum_{\substack{c' \\ \delta_{c'} > 0}} 1$$

Now $\sum_{c} \mathbb{E}(\Delta_{\theta_c} \phi)$

$$\leq \sum_{\substack{c' \\ \delta_{c'} = 0}} (-1) + \sum_{\substack{c' \\ \delta_{c'} > 0}} \frac{11}{6}\delta_c - 1$$

$$= \left(k - \sum_{\substack{c' \\ \delta_{c'} > 0}} 1 \right)(-1) + \sum_{\substack{c' \\ \delta_{c'} > 0}} \frac{11}{6}\delta_c - 1$$

$$= -k + \sum_{\substack{c' \\ \delta_{c'} > 0}} \frac{11}{6} \delta_c$$

$$= -k + \frac{11}{6} \delta$$

$\therefore$ If $k > \frac{11}{6} \Delta$, this will be negative.

Proof of Main Lemma

a) If $\delta_c = 0$, $\Rightarrow D_c = \{ S_\sigma(v,c)$
$S_\tau(v,c) \}$

Also $S_\sigma(v,c) = S_\tau(v,c)$
$= \{v\}$.

Thus the only possible coupling is identity. That reduces distance by 1.

$\Rightarrow \mathbb{E}[\Delta_{D_c} \phi] = -1$

b) $$S_\sigma(v,c) = \bigcup_i S_\tau(\overset{c \text{ colored neighbors}}{w_i}, \sigma(v)) \cup \{v\}$$

$\downarrow$

v "blocks" the clusters.

Similarly –

$$S_\tau(v,c) = \bigcup_j S_\sigma(w_j, \tau(v)) \cup \{v\}.$$

<u>Let</u>

$a_i = |S_\tau(w_i, \sigma(v))|$

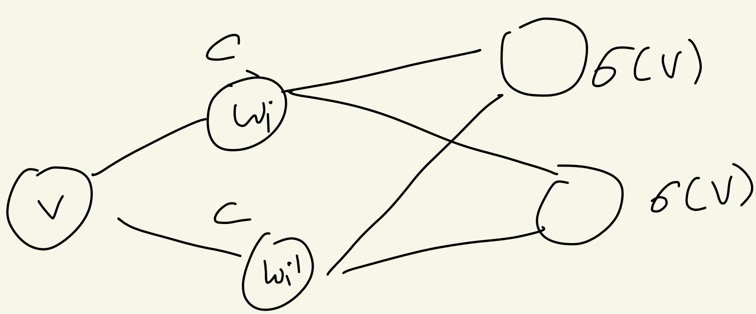
$A = |S_\sigma(v, c)|$

$b_j = |S_\sigma(w_j, \tau(v))|$

$B = |S_\tau(v, c)|$

<u>Technicality</u>: It may be that

$$S_\tau(w_i, \sigma(v)) = S_\tau(w_{i'}, \sigma(v))$$



For such cases just set a_i = 0. i.e. each such cluster is indexed by exactly 1 neighbor.

Coupling –

Let

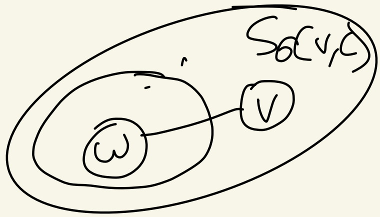
$$a_{max} = \max_i a_i$$

$$i_{max} = \underset{i}{\arg\max}\, a_i$$

$$b_{max} = \max_j b_j$$

$$j_{max} = \underset{j}{\arg\max}\, b_j$$

Couple:



1. With prob p_A flip $S_\sigma(v,c)$ and $S_\tau(\omega_{i_{max}}, \sigma(v))$

2. With prob p_B flip $S_\tau(v,c)$ and $S_\sigma(\omega_{j_{max}}, \tau(v))$

3. Let

$$q_l = \begin{cases} p_{a_l} - p_A, & \text{if } l = i_{max} \\ p_{a_l}, & \text{o/w} \end{cases}$$

$$q'_l = \begin{cases} p_{b_l} - p_B, & \text{if } l = j_{max} \\ p_{b_l}, & \text{o/w} \end{cases}$$

For each w_ℓ,

3a. With prob $\min(q_\ell, q'_\ell)$ flip $S_\tau(w_\ell, \sigma(v))$ and $S_\sigma(w_\ell, \tau(v))$

3b. With prob $q_\ell - \min(q_\ell, q'_\ell)$ flip $S_\tau(w_\ell, \sigma(v))$

3c. With prob $q'_\ell - \min(q_\ell, q'_\ell)$ flip $S_\sigma(w_\ell, \tau(v))$

Marginal: a) $S_\sigma(v,c)$ is flipped with prob P_A

$S_\tau(v,c)$ is flipped with prob P_B

in step 1/2

b) $S_\sigma(w_{j_{max}}, \tau(v))$

WLOG assume $q_\ell < q'_\ell$

flipped with prob P_A in step 1.

Then with prob $P_{a_\ell} - P_A$ in step 2 a

$\therefore$ totally with prob P_{a_ℓ}.

c) $S_t(W_{imax}, 6(v))$

WLOG assume $q_\ell < q_{\ell'}$

$\Rightarrow$ flipped with prob P_B in step 2.

with prob q_ℓ in step 3a.

with prob $P_{b\ell} - P_B - q_\ell$

$\therefore$ totally with prob

$$P_B + q_\ell + P_{b\ell} - P_B - q_\ell$$

$$= P_{b\ell}.$$

Similarly for the others.

Thus each cluster is flipped

with prob p_α.

Coupling Analysis

Step 1: The nodes shared b/w $S_\sigma(v, c)$ and $S_\tau(w_{i_{max}}, \sigma(v))$, are identical.

$\Rightarrow$ Distance increases by at most $A - a_{max} - 1$

↳ v was already mismatched

Step 2: Similarly,

$B - b_{max} - 1$.

Step 3a: $S_\tau(w_\ell, \sigma(v))$ and $S_\sigma(w_\ell, \tau(v))$ are disjoint apart from w_ℓ. $\therefore$ dist increases by $a_\ell + b_\ell - 1$

w_ℓ counted twice.

Step 3b: Dist increases by a_ℓ

3c: Dist increases by b_ℓ

Let $f(w_\ell) = \min(q_\ell, q'_\ell) \cdot (a_\ell + b_\ell - 1)$

$+ (q_\ell - \min) a_\ell + (q'_\ell - \min) b_\ell$

$$= q_\ell a_\ell + q'_\ell b_\ell - \min$$

Now $\mathbb{E}[\Delta_{D_c}\Phi] \leq P_A(A - a_{max} - 1)$

$$+ P_B(B - b_{max} - 1) + \sum_{\omega_\ell \in \Gamma'_c} f(\omega_\ell)$$

Cases

$\underline{\delta_c = 1}$

Here $A = a_1 + 1$, $B = b_1 + 1$

WLOG: $q_1 \geq q'_1$

$\Rightarrow f(\omega_1) = q_1 a_1 + q'_1 b_1 - q'_1$

$$\Rightarrow \mathbb{E}(\Delta_{D_c}\Phi) \leq P_A(a_1 + 1 - a_1 - 1) + P_B(b_1 + 1 - b_1 - 1)$$
$$+ q_1 a_1 + q'_1 b_1 - q'_1$$

substituting $q_1 = p_{a_1} - p_A$

$$q'_1 = p_{b_1} - p_B$$

$$\Rightarrow \mathbb{E}(\Delta_{PC}\Phi) \leq (p_{a_1} - p_A)\, a_1 + (p_{b_1} - p_B)\, b_1 - (p_{b_1} - p_B)$$

$$\leq a_1 (p_{a_1} - p_A) + (b_1 - 1)(p_{b_1} - p_B)$$

$$\leq a_1 (p_{a_1} - p_{a_1+1}) + (b_1 - 1)(p_{b_1} - p_{b_1+1})$$

Recall "key property"

$$i(p_i - p_{i+1}) + (j-1)(p_j - p_{j+1}) \leq \frac{5}{6}$$

$$\Rightarrow \mathbb{E}(\Delta_{D_c} \Phi) \leq \frac{5}{6}$$

$\underline{\delta_c = 2}$

Claim: When $\delta_c = 2$, $\mathbb{E}[\Delta_{D_c} \Phi]$ is maximized (worst case) for

$$a_1 = a_2 = a \leq 3 \quad \text{and}$$

$$b_1 = b_2 = b = 1$$

Given the claim –

$$f(\omega_1) = a\, p_a + b\, p_b - p_a$$

$$f(\omega_2) = a(p_a - p_A) + b(p_b - p_B) - (p_a - p_A)$$

$$\mathbb{E}[\Delta_{D_c}\phi] \leq p_A(A-a-1) + p_B(B-b-1)$$
$$+ (a-1)p_a + b\,p_b$$
$$+ (a-1)(p_a - p_A) + b(p_b - p_B)$$

Substituting $A = 2a+1$
$B = 2b+1 = 3$

$$\leq p_{2a+1}(2a+1-a-1) + p_{2b+1}(2b-b)$$
$$+ (a-1)(2p_a - p_{2a+1}) + b(2p_b - p_B)$$

$$\leq 2(a-1)p_a + p_{2a+1} + 2b\,p_b$$

$$\leq 2(a-1)p_a + p_{2a+1} + 2 \left[\because\ p_1 = 1\right]$$

First "key property"

$$2(i-1)p_i + p_{2i+1} \leq \frac{2}{3}$$

$$\Rightarrow \mathbb{E}[\Delta_{DC}\Phi] \leq \frac{2}{3} + 2$$

$$= \frac{2}{3} + 3 - 1$$

$$= \frac{4+18}{6} - 1$$

$$= 2\left(\frac{2+9}{6}\right) - 1$$

$$= \delta_c\left(\frac{11}{6}\right) - 1$$

$\underline{\delta_c > 2}$

Let $g(\omega_l) = a_l p_{a_l} + b_l p_{b_l}$
$- \min(p_{a_l}, p_{b_l})$

Note: $g(\omega_l) = f(\omega_l)$
for $l \neq i_{max} \neq j_{max}$.

If $l = i_{max} = j_{max}$

$$f(\omega_l) = a_{max}(p_{a_{max}} - p_A) + b_{max}(p_{b_{max}} - p_B) - \min(p_{a_{max}} - p_A, p_{b_{max}} - p_B)$$

$$\leq a_{max}(P_{max} - P_A) + b_{max}(P_{bmax} - P_B)$$

$$- \min(P_{amax}, P_{bmax})$$

$$+ P_A + P_B$$

$$= g(w_e) + P_A(1 - a_{max})$$

$$+ P_B(1 - b_{max})$$

If $i_{max} \neq j_{max}$

$$f(w_{imax}) + f(w_{jmax}) =$$

$$a_{max}(P_{amax} - P_A) + b_i P_{bi}$$

$$- \min(P_{amax} - P_A, P_{bi}) + a_j P_{aj}$$

$$+ b_{max}(P_{bmax} - P_B) - \min(P_{aj}, P_{bmax} - P_B)$$

$$\leq a_{max} p_{a_{max}} + b_i p_{b_i} - \min(p_{a_{max}}, p_{b_i}) + p_A + b_{max} p_{b_{max}} + a_j p_{a_j} - \min(p_{b_{max}}, p_{a_j}) + p_B$$

$$= a_{max} p_A - b_{max} p_B$$

$$\leq g(\omega_{i_{max}}) + g(\omega_{j_{max}}) + p_A(1 - a_{max}) + p_B(1 - b_{max})$$

$$\Rightarrow \sum_{l} f(\omega_l) \leq \sum_{l} g(\omega_l) + p_A(1 - a_{max}) + p_B(1 - b_{max})$$

$$\Rightarrow \mathbb{E}(\Delta_{D_c}\phi) \leq p_A(A - a_{max} - 1) + p_B(B - b_{max} - 1) + \sum_{l} g(\omega_l) + p_A(1 - a_{max}) + p_B(1 - b_{max})$$

$$\leq p_A(A - 2a_{max}) + p_B(B - 2b_{max}) + \sum_{l} g(\omega_l)$$

Recall "key property"

$$(i-m)\,p_i < \frac{1}{4} \quad \text{for } m \geq 2$$

here $2a_{max} \geq 2$

hence $p_A(A - 2a_{max}) < \frac{1}{4}$

$$p_B(B - 2b_{max}) < \frac{1}{4}$$

$$g(w_\ell) = a_\ell p_{a\ell} + b_\ell p_{b\ell} - \min(p_{a\ell}, p_{b\ell})$$

wloa: $a_\ell \leq b_\ell$ and hence

$$p_{a\ell} \geq p_{b\ell}$$

$$\Rightarrow g(w_\ell) = a_\ell p_{a\ell} + (b_\ell - 1)\, p_{b\ell}$$

$$\left.\begin{array}{l} a_\ell p_\ell \le \overbrace{1 \cdot p_1}^{\to 1} \\ (b_\ell - 1) p_{b_\ell} \le \underbrace{2 p_3}_{\to 1/3} \end{array}\right\} \text{"Key properties"}$$

$$\Rightarrow g(w_\ell) \le \frac{4}{3}$$

$$\Rightarrow \mathbb{E}(\Delta_{D_c} \phi) \le \frac{1}{4} + \frac{1}{4} + \sum_\ell \frac{4}{3}$$

$$\le \frac{1}{2} + \delta_c \cdot \frac{4}{3}$$

$$\le \frac{11}{6} \delta_c - 1 \;, \quad \delta_c > 2$$